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Given two topologies P and Q for a set X, this paper examines the upper bound topology T[P,Q]. The objective is to determine what topological properties are inherited by T[P,Q] from P and Q and if T[P,Q] has certain topological properties then when must P and Q have these properties.

TOPOLOGICAL PROPERTIES RELATED

TO THE UPPER BOUND

TOPOLOGY

by

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INTRODUCTION AND PRELIMINARY REMARKS

If X is a set with topologies S and T, the upper bound topology for X is the set $T[S,T]$ defined as follows: a set $U \subset X$ is in $T[S,T]$ provided if $x \in U$ there is an $s \in S$ and a $t \in T$ such that $x \in s \cap t \subset U$. In this paper we examine the upper bound topology and its relation to the topologies S and T.

In Chapter I we look at the separation axioms. We find that if (X,S) is T_0, T_1, or Hausdorff, then $(X,T[S,T])$ has the same property. It is also observed that the implication is not reversible. Further, it is shown that if $(X,T[S,T])$ is either regular, normal, Urysohn, or metric then no conclusion can be drawn about (X,S).

In Chapter II we look at countability and find that if (X,S) and (X,T) are both first countable or second countable then $(X,T[S,T])$ has the same property. It is also seen that if $(X,T[S,T])$ is separable then (X,S) is separable. Examples are given to show that the implications are not reversible.

In Chapter III it is observed that if $(X,T[S,T])$ is either compact, countably compact, Lindelof, or connected then so is (X,S). Examples are given to show that no conclusion can be drawn about locally connected.

The reader is expected to have a working knowledge of point-set topology and is referred to [1], [2], and [3] for definitions and theorems not covered in this paper.

CHAPTER I

SEPARATION

DEFINITION 1. If X is a set with topologies S and T the set $T[S,T]$ is defined as follows: if $U \subset X$ then $U \in T[S,T]$ provided if $x \in U$ then there is an $s \in S$ and a $t \in T$ such that $x \in s \cap t \subset U$.

THEOREM 1. If X is a set with topologies S and T, then $T[S,T]$ is a topology for X.

PROOF: Let $x \in X$. Then $x \in X \cap X \subset X$ and X is in both S and T so $X \in T[S,T]$. Clearly $\phi \in T[S,T]$. Let U and V be in $T[S,T]$ and let $x \in U \cap V$. Since $x \in U$ and $x \in V$ there exist sets s and m in S and t and n in T such that $x \in s \cap t \subset U$ and $x \in m \cap n \subset V$. Then $x \in s \cap m$ and $x \in t \cap n$ and since S and T are topologies, $s \cap m \in S$ and $t \cap n \in T$. It follows that $x \in (s \cap m) \cap (t \cap n) \subset U \cap V$ and that $U \cap V \in T[S,T]$.

Now let A be an index set and let $\{W_\alpha \mid \alpha \in A\} \subset T[S,T]$ and let $x \in \cup\{W_\alpha \mid \alpha \in A\}$. Then there is an $\alpha_1 \in A$ such that $x \in W_{\alpha_1}$. But $W_{\alpha_1} \in T[S,T]$ so there are sets $s \in S$ and $t \in T$ such that $x \in s \cap t \subset W_{\alpha_1} \subset \cup\{W_\alpha \mid \alpha \in A\}$. Therefore $\cup\{W_\alpha \mid \alpha \in A\} \in T[S,T]$ and $T[S,T]$ is a topology for X.

THEOREM 2. Let X be a set with topologies S and T. Then $T[S,T]$ is the smallest topology for X containing S and T.

<u>PROOF</u>: Let $U \in S$ and $x \in U$. Then since $X \in T$ and $x \in U \cap X \subset U$, we have $U \in T[S,T]$. Therefore $S \subset T[S,T]$ and similarly $T \subset T[S,T]$.

Now suppose there is another topology Γ for X such that $S \subset \Gamma$, $T \subset \Gamma$, and $\Gamma \subsetneq T[S,T]$. Then there is a set $U \in T[S,T]$ such that $U \notin \Gamma$. Let $x \in U$. Then there are sets $s \in S$ and $t \in T$ such that $x \in s \cap t \subset U$. But $s \in \Gamma$ and $t \in \Gamma$ so $s \cap t \in \Gamma$ and therefore $U \in \Gamma$. This contradicts the assumption that $U \notin \Gamma$, so the topology Γ does not exist and $T[S,T]$ is the smallest topology containing S and T.

<u>DEFINITION 2</u>. Let X be a set. Then $U \subset X$ is in the <u>cofinite topology for</u> X provided the complement of U is finite or $U = X$ or $U = \phi$.

<u>DEFINITION 3</u>. If X is a set, let every subset of X be open. The resulting topology is called the <u>discrete topology for</u> X.

<u>DEFINITION 4</u>. If X is a set and $p \in X$, the <u>particular point topology for</u> X <u>with respect to</u> p is defined as follows: a set V is open if $p \in V$ or if $V = \phi$ or $V = X$.

Clearly the sets mentioned in definitions 2, 3, and 4 are topologies for X.

<u>EXAMPLE 1</u>. Let X be the real numbers and let T be the cofinite topology for X. Let $p \in X$ and let S be the particular point topology with respect to p. Then there is a set $U \in T[S,T]$ such that $U \notin S$ and $U \notin T$.

PROOF: Let $x \in X$ such that $x \neq p$ and let $t = X - \{p\}$ and $s = \{x\} \cup \{p\}$. Then $t \in T$ and $s \in S$ and $x \in s \cap t \subset \{x\}$. Thus $\{x\} \in T[S,T]$ but $\{x\} \notin S$ and $\{x\} \notin T$.

DEFINITION 5. Let X be a set and T a topology for X. Then if $U \subset X$, by U is T-open it is meant that $U \in T$ and by U is T-closed it is meant that $X - U \in T$.

DEFINITION 6. A topological space X is a T_0-space if and only if whenever x and y are distinct points in X, there is an open set containing one and not the other.

THEOREM 3. Let (X,S) and (X,T) be topological spaces with the property that (X,S) is a T_0-space. Then $(X,T[S,T])$ is a T_0-space.

PROOF: Let x and y be distinct points of X. Then there is a set $s \in S$ which contains one of x and y but not the other. Let $t \in T$ such that both x and y are in t and let $U = s \cap t$. Clearly $U \in T[S,T]$ and U contains one of x and y but not the other. Therefore $(X,T[S,T])$ is a T_0-space.

DEFINITION 7. Let X be the real numbers and define the usual topology for X as follows: a subset U of X is open provided if $p \in U$ there is a positive number c such that $(p-c,p+c) \subset U$.

EXAMPLE 2. Let $X = (0,1)$ and define topologies S and T for X as follows: $S = (\psi - \{U \in \psi \mid 1/4 \in U \text{ or } 1/2 \in U\}) \cup \{X\}$ and $T = (\psi - \{U \in \psi \mid 7/8 \in U \text{ or } 3/4 \in U\}) \cup \{X\}$ where ψ is the usual topology for X. Then $(X,T[S,T])$ is a T_0-space, but (X,S) and (X,T) are not.

PROOF: Consider the points 1/4 and 1/2 in (X,S). The only S-open set containing 1/4 is X, but X also contains 1/2. Similarly, the only S-open set containing 1/2 also contains 1/4. Therefore (X,S) is not a T_0-space, nor is (X,T).

Now since T[S,T] is the smallest topology containing S and T, T[S,T] = ψ. Let x and y be distinct points in X and let $c = |x - y|/2$. Then $(x - c, x + c)$ is an open set containing x but not y. Therefore (X,T[S,T]) is a T_0-space.

DEFINITION 8. A topological space X is a T_1-space if and only if whenever x and y are distinct points in X, there is an open set about each one which does not contain the other.

THEOREM 4. If (X,S) and (X,T) are topological spaces and if (X,S) is a T_1-space, then (X,T[S,T]) is a T_1-space.

PROOF: Let x and y be distinct points in X. Then there are open sets s_1 and s_2 in S such that $x \in s_1$ but $y \notin s_1$ and $y \in s_2$ but $x \notin s_2$. Let $t \in T$ such that $x,y \in t$ and let $U_1 = s_1 \cap t$ and $U_2 = s_2 \cap t$. Then $U_1, U_2 \in T[S,T]$ such that $x \in U_1$ but $y \notin U_1$ and $y \in U_2$ but $x \notin U_2$. Therefore (X,T[S,T]) is a T_1-space.

DEFINITION 9. If (X,T) is a topological space and $A \subset X$, the collection $T' = \{G \cap A \mid G \in T\}$ is a topology for A called the relative topology for A. The fact that a subset of X is being given this topology is signified by referring to it as a subspace of X.

<u>EXAMPLE 3</u>. Let $X = [0,1]$ be a subspace of the reals with the usual topology and let T' denote the relative topology for X. Define topologies S and T for X as follows:

$S = (T' - \{U \mid U \in T' \text{ and } 0 \in U\}) \cup \{X\}$ and

$T = (T' - \{U \mid U \in T' \text{ and } 1 \in U\}) \cup \{X\}$. Then $(X, T[S,T])$ is a T_1-space but (X,S) and (X,T) are not.

<u>PROOF</u>: By theorem 2, $T[S,T] = T'$. Let x and y be distinct points of X and let $c = |x - y|/4$. Then $(x - c, x + c)$ is a $T[S,T]$-open set containing x but not y, and $(y - c, y + c)$ is a $T[S,T]$-open set containing y but not x. Therefore $(X,T[S,T])$ is a T_1-space.

Now consider the points 0 and 1 in X. The only S-open set containing 0 also contains 1 and the only T-open set containing 1 also contains 0. Therefore neither (X,S) nor (X,T) is a T_1-space.

<u>DEFINITION 10</u>. A topological space X is a <u>Hausdorff space</u> if and only if whenever x and y are distinct points of X, there are disjoint open sets U and V in X with $x \in U$ and $y \in V$.

<u>THEOREM 5</u>. If X is a set and S and T are topologies for X such that (X,S) is Hausdorff, then $(X,T[S,T])$ is Hausdorff.

<u>PROOF</u>: Let x and y be distinct points of X. Then there are disjoint S-open sets U and V in X such that $x \in U$ and $y \in V$. By theorem 2, U and V are also $T[S,T]$-open so $(X,T[S,T])$ is Hausdorff.

EXAMPLE 4. Let X, S, and T be defined as in example 3. Then $(X,T[S,T])$ is Hausdorff, but (X,S) and (X,T) are not.

PROOF: Let x and y be distinct points in X and let $c = |x - y|/4$ as in example 3. Then $(x - c, x + c)$ and $(y - c, y + c)$ are disjoint $T[S,T]$-open sets containing x and y, respectively. Therefore $(X,T[S,T])$ is a Hausdorff space. Since (X,S) and (X,T) are not T_1-spaces, they clearly cannot be Hausdorff.

DEFINITION 11. A topological space X is a regular space provided whenever A is closed in X and $x \notin A$, then there are disjoint open sets U and V with $x \in U$ and $A \subset V$.

DEFINITION 12. Let X be a set and let $p \in X$. We define the excluded point topology for X with respect to p by declaring open, in addition to X, all sets which do not include the point p.

EXAMPLE 5. Let X be the real numbers and let S be the particular point topology for X with respect to the point 1 and T, the excluded point topology for X with respect to 1. Then $(X,T[S,T])$ is regular, but (X,S) and (X,T) are not.

PROOF: The topology S cannot be regular since there are no disjoint open sets. Now consider the T-closed set $A = (0,1]$ and the point 2. The only T-open set containing 1 is X and $2 \in X$, so T is not regular.

Clearly $T[S,T]$ is the discrete topology for X. Let $A \subset X$ and $x \notin A$. Since every set is both open and closed, A and $\{x\}$ are disjoint open sets containing A and x, respectively. Therefore $(X,T[S,T])$ is regular.

DEFINITION 13. A topological space X is normal provided whenever A and B are disjoint closed sets in X, there are disjoint open sets U and V with $A \subset U$ and $B \subset V$.

EXAMPLE 6. Let X, S, and T be defined as in example 5. Then (X,T[S,T]) is normal, but (X,S) is not.

PROOF: Clearly T[S,T] is the discrete topology. Let A and B be disjoint closed sets. Then since A and B are also open we have that (X,T[S,T]) is normal.

Since any S-open set must contain the point 1 there can be no disjoint S-open sets and therefore the space (X,S) is not normal.

The author believes there are examples of spaces (X,S) and (X,T) which are normal while (X,T[S,T]) is not normal and similar examples of regular spaces, but to this date has been unable to find any.

DEFINITION 14. A topological space X is a Urysohn space provided whenever x and y are distinct points in X there are open sets U containing x and V containing y such that $\overline{U} \cap \overline{V} = \phi$.

EXAMPLE 7. Let X be the real numbers and let S be the particular point topology with respect to 1 and T, the excluded point topology with respect to 1. Then (X,T[S,T]) is a Urysohn space, but (X,S) and (X,T) are not.

PROOF: Since every point in X except 1 is a limit point of 1 under the topology S, the closure of any open set is X.

Thus (X,S) is not Urysohn. Now since every T-closed set contains the point 1 there are no disjoint T-closed sets and (X,T) is not Urysohn.

Clearly $T[S,T]$ is the discrete topology for X. Let x and y be distinct points in X. Then $\{x\}$ and $\{y\}$ are open sets containing x and y, respectively, and since $\overline{\{x\}} = \{x\}$ and $\overline{\{y\}} = \{y\}$, $\overline{\{x\}} \cap \overline{\{y\}} = \phi$. Therefore $(X,T[S,T])$ is a Urysohn space.

DEFINITION 15. Let X be a set and let $\P$ be a function mapping $X \times X$ into the positive real numbers such that each of the following holds:

(i) $\P(x,y) = 0$ if and only if $x = y$ for every $x,y \in X$,

(ii) $\P(x,y) = \P(y,x)$ for every $x,y \in X$, and

(iii) $\P(x,z) \leq \P(x,y) + \P(y,z)$ for every $x,y,z \in X$.

Then $\P$ is said to be a metric for X and $(X,\P)$ is a metric space.

DEFINITION 16. Let X be a set and let $\P$ be a metric for X. Define the metric topology $T_{\P}$ for X as follows: a set $U \subset X$ is open provided if $x \in U$, there is a positive real number ε such that $x \in \{y \mid \P(x,y) < \varepsilon\} \subset U$.

DEFINITION 17. Let (X,T) be a topological space. If there is a metric $\P$ for X such that $T = T_{\P}$, then (X,T) is said to be metrizable.

THEOREM 6. Let (X,T) be a metrizable topological space. Then (X,T) is Hausdorff.

PROOF: Let $\P$ be the metric for which $T = T_{\P}$. Let x and y be two distinct points of X and let $\zeta = \P(x,y)/2$. Let $U = \{z \mid \P(x,z) < \zeta\}$ and let $V = \{r \mid \P(y,r) < \zeta\}$. Then U and V are in $T = T_{\P}$, $U \cap V = \phi$, $x \in U$ and $y \in V$. Therefore (X,T) is Hausdorff.

EXAMPLE 8. Let X be the reals and let S be the particular point topology with respect to the point 1 and T, the excluded point topology with respect to 1. Then $(X,T[S,T])$ is metrizable, but (X,S) and (X,T) are not.

PROOF: Since $T[S,T]$ is the smallest topology containing both S and T, it is clear that $T[S,T]$ is the discrete topology. Now define the metric $\P$ by $\P(x,y) = 1$ if $x \neq y$ and $\P(x,y) = 0$ if $x = y$. Clearly $\P$ is a metric for X. Let $x \in X$ and let $\zeta = 1/2$. Then $x \in \{y \mid \P(x,y) < 1/2\} \subset \{x\}$. Therefore $(X,T[S,T])$ is metrizable.

Since there are no disjoint open sets in (X,S) the space is not Hausdorff and therefore is not metrizable. Consider the point 1 in (X,T) and let x be another point in (X,T). The only T-open set containing 1 is X, but $x \in X$ so (X,T) is not Hausdorff and thus not metrizable.

CHAPTER II

COUNTABILITY

DEFINITION 18. If X is a topological space and $x \in X$, a neighborhood of x is a set U which contains an open set containing x. The collection U_x of all neighborhoods of x is the neighborhood system at x.

DEFINITION 19. A neighborhood base at x in the topological space X is a subcollection B_x taken from the neighborhood system U_x, having the property that each $U \in U_x$ contains some $V \in B_x$. Elements of the neighborhood base at x are called basic neighborhoods at x.

DEFINITION 20. A set A is countable if and only if there is a one-to-one and onto function from A to the positive integers.

DEFINITION 21. A topological space X is first countable if and only if each $x \in X$ has a countable neighborhood base.

THEOREM 7. Let X be a set with topologies S and T such that (X,S) and (X,T) are first countable. Then $(X, T[S,T])$ is first countable.

PROOF: Let $x \in X$ and let $U \in T[S,T]$ such that $x \in U$. Then there is an $s \in S$ and a $t \in T$ such that $x \in s \cap t \subset U$. There are countable neighborhood bases B_x^S and B_x^T at x for S and T, respectively. So there is an $M \in B_x^S$ and an $N \in B_x^T$

such that $x \in M \subset s$ and $x \in N \subset t$. Then $x \in M \cap N \subset s \cap t \subset U$ and $B_x^{T[S,T]} = \{M \cap N \mid M \in B_x^S \text{ and } N \in B_x^T\}$ is a neighborhood base at x in $T[S,T]$. Furthermore, since B_x^S and B_x^T are both countable, $B_x^{T[S,T]}$ is countable.

<u>EXAMPLE 9</u>. Let X be the real numbers and let S be the cofinite topology and T, the usual topology. Then $(X,T[S,T])$ is first countable, but (X,S) is not.

<u>PROOF</u>: Let $U \in S$. Then $X - U$ is finite so $U = X - \{a_1, a_2, \ldots, a_n \mid a_i \in R \text{ for } i = 1, 2, \ldots, n\}$, where R is the set of real numbers. Given any $p \in U$ it is clear that one can find a real number c such that $(p - c,p + c) \subset U$. It follows that $S \subset T$ and $T[S,T] = T$.

Now let $x \in X$ and $U \in T$ such that $x \in U$. By definition of T there is a number p such that $(x - p,x + p) \subset U$. There is a rational number c between x and p so $(x - c,x + c) \subset (x - p,x + p) \subset U$ and the set $B_x = \{(x - c,x + c) \mid c \text{ is rational}\}$ forms a neighborhood base at x of elements from T. Since the set of rationals is countable, B_x is countable and $T[S,T] = T$ is first countable.

Suppose there is a countable neighborhood base at x of elements from S and denote it B'. Now every open set containing x contains some $B \in B'$ so $\cap B' = \{x\}$. Therefore $X - \{x\} = X - \cap B' = \cup\{X - B \mid B \in B'\}$ and since each $B \in B'$ is S-open $(X - B)$ is finite. The countable union of finite sets is countable so $X - \{x\}$ is countable. But since the set of real numbers is uncountable this is a contradiction and (X,S) is not first countable.

DEFINITION 22. Let X be a set. The trivial topology for X is the topology having X and ϕ as the only open sets.

EXAMPLE 10. Let X be the real numbers and let S be the cofinite topology and T the trivial topology. Then (X,T) is first countable, but $(X,T[S,T])$ is not.

PROOF: Let $x \epsilon X$. The only T-open set containing x is X, so X is a countable neighborhood base at x. Since $T \subset S$, $T[S,T] = S$ by theorem 2 and by example 9 S is not first countable.

DEFINITION 23. If (X,T) is a topological space, the collection B° is called a base for (X,T) if and only if whenever G is an open set in X and $p \epsilon G$, there is a set $B \epsilon B^\circ$ such that $p \epsilon B \subset G$.

DEFINITION 24. A topological space X is second countable provided its topology has a countable base.

THEOREM 8. Let X be a set and let S and T be topologies for X. If (X,S) and (X,T) are second countable then $(X,T[S,T])$ is second countable.

PROOF: Since (X,S) and (X,T) are second countable, S and T have countable bases B_S and B_T, respectively. Let $B' = \{U \cap V \mid U \epsilon B_S \text{ and } V \epsilon B_T\}$. Also let $G \epsilon T[S,T]$ and $p \epsilon G$. Then there are sets $s \epsilon S$ and $t \epsilon T$ such that $p \epsilon s \cap t \subset G$. By the definition of a base, there are sets $U \epsilon B_S$ and $V \epsilon B_T$ such that $p \epsilon U \subset s$ and $p \epsilon V \subset t$. Then

$p \in U \cap V \subset s \cap t \subset G$ and B' is a base for the topology $T[S,T]$.

Clearly, since B_S and B_T are countable, B' is countable and therefore $(X,T[S,T])$ is second countable.

<u>LEMMA 1</u>. Let (X,S) be a topological space with the property that X is second countable. Then X is first countable.

<u>PROOF</u>: Since X is second countable the topology S has a countable base B_S. Let $x \in X$ and let $B_x = \{B \mid B \in B_S \text{ and } x \in B\}$. Also let $U \in S$ such that $x \in U$. Since B_S is a base for S, there is some $B \in B_S$ such that $x \in B \subset U$. But $x \in B$ implies that $B \in B_x$ so B_x is a neighborhood base at x. Clearly B_x is countable since B_S is. Therefore the space (X,S) is first countable.

<u>EXAMPLE 11</u>. Let X be the real numbers and let S be the cofinite topology for X and T the usual topology. Then $(X,T[S,T])$ is second countable, but (X,S) is not.

<u>PROOF</u>: Let $B_T = \{(a,b) \mid a \text{ and } b \text{ are rational numbers}\}$ and let $x \in U \in T$. There are rational numbers a and b such that $x \in (a,b) \subset U$, so B_T is a base for x. The set B_T is certainly countable since the set of rational numbers is countable. It has been shown in example 9 that $T[S,T] = T$ so we have that $(X,T[S,T])$ is second countable. By the same example we know that (X,S) is not first countable and therefore by lemma 1, (X,S) is not second countable.

DEFINITION 25. A set D is dense in a topological space (X,S) if and only if the closure of D in X is X.

DEFINITION 26. A topological space (X,S) is separable provided X has a countable dense subset.

THEOREM 9. Let X be a set and S and T topologies for X. If $(X,T[S,T])$ is separable, then (X,S) is separable.

PROOF: Since $(X,T[S,T])$ is separable, X has a oountable dense subset D. Then $\bar{D}^{T[S,T]} = X$. Let $x \in X$ such that $x \notin D$ and let U be an S-open set containing x. Since $U \in S$, we know that $U \in T[S,T]$ and since $\bar{D}^{T[S,T]} = X$, U must contain some point $d \in D$ such that $d \neq x$. Then x is an S-limit point of D and therefore $\bar{D}^{S} = X$ and (X,S) is separable.

EXAMPLE 12. Let X be the real numbers and let S be the topology having the set $\{[a,b) \mid a < b,\ a,b \in X\}$ as a base. Let T be the topology having the set $\{(a,b] \mid a < b,\ a,b \in X\}$ as a base. Then (X,S) and (X,T) are separable, but $(X,T[S,T])$ is not.

PROOF: Let $\mathbf{Q}$ be the set of rational numbers and let $x \in X - \mathbf{Q}$. Now if $[a,b) \in S$ such that $x \in [a,b)$ there is a rational number q between x and b and thus $q \in [a,b)$. Therefore $\bar{\mathbf{Q}}^{S} = X$. It follows clearly from this that $\bar{\mathbf{Q}}^{T} = X$ and therefore that (X,S) and (X,T) are separable.

Now let $x \in X$ and let $a,b \in X$ such that $a < x < b$. Then $(a,x] \in T$ and $[x,b) \in S$ such that $x \in (a,x] \cap [x,b) \subset \{x\}$.

Therefore $\{x\} \in T[S,T]$ and $T[S,T]$ is the discrete topology.

Suppose that $(X,T[S,T])$ has a countable dense subset D. Let $x \in X - D$. Then every $T[S,T]$-open set containing x must also contain some $d \in D$ such that $d \neq x$. The set $\{x\}$ is a $T[S,T]$-open set containing x, but there is no $d \in D$ such that $d \in \{x\}$. This is a contradiction and so $(X,T[S,T])$ is not separable.

<u>EXAMPLE 13</u>. Let X, S, and T be defined as in example 12. Then (X,S) and (X,T) are Hausdorff and separable, but $(X,T[S,T])$ is not separable.

<u>PROOF</u>: Let $x,y \in X$. Without loss of generality we may assume that x is less than y. Then for any $b \in X$ such that $y < b$, $[x,y)$ and $[y,b)$ are disjoint S-open sets containing x and y, respectively, and for any $a \in X$ such that $a < x$, $(a,x]$ and $(x,y]$ are disjoint T-open sets about x and y, respectively. Therefore (X,S) and (X,T) are Hausdorff. The rest of the example follows from example 12.

<u>DEFINITION 27</u>. A separable topological space (X,S) is <u>hereditarily separable</u> provided every subspace of X is separable.

<u>EXAMPLE 14</u>. Let $X = \{(x,y) \mid x \text{ and } y \text{ are real numbers and } y \geq 0\}$. Define the topology S for X as follows: if $(x,y) \in X$ and $y > 0$, a basic neighborhood at (x,y) will be the usual disc with center (x,y) and if $y = 0$, a basic neighborhood at (x,y) will be the union of the semicircle having center (x,y) with its interior,

excluding all those points on the x-axis except the point (x,y). Then (X,S) is separable but not hereditarily separable.

PROOF: If $\mathbf{Q}$ is the set of rational numbers let $D = \{(p,q) \mid p,q \in \mathbf{Q} \text{ and } q \geq 0\}$. Since the set of rationals is countable, D is clearly countable. Now let $(x,y) \in X - D$ and let U be an S-open set containing (x,y). There must be an ordered pair $(p,q) \in D$ such that $(p,q) \in U$ so (x,y) is a limit point of D. Therefore D is dense in (X,S) and (X,S) is separable.

Now let $A = \{(x,0) \mid x \in \text{Reals}\}$ and let $(x,0) \in A$. Let $U \in S$ such that $(x,0) \in U$. Then $(x,0) = \{(x,0)\} \cap U$ and therefore the relative topology S_A for A is the discrete topology. Suppose D' is a countable dense subset of (A,S_A) and consider the point $(y,0) \in A - D'$. Then every S_A-open set containing $(y,0)$ must contain a point of A different from $(y,0)$. However, $\{(y,0)\}$ is an S_A-open set which contains no point of $A - D'$, a contradiction. Therefore (A,S_A) is not separable.

CHAPTER III

COMPACTNESS AND CONNECTEDNESS

DEFINITION 28. A cover of a space X is a collection A of subsets of X whose union is all of X. A subcover of a cover A is a subcollection A' of A which is a cover. An open cover of X is a cover consisting only of open sets.

DEFINITION 29. A topological space X is compact if and only if each open cover of X has a finite subcover.

THEOREM 10. Let X be a set and let S and T be topologies for X. If $(X,T[S,T])$ is compact then (X,S) is compact.

PROOF: Let I be an index set and let $A = \{U_i \mid i \in I\}$ be an S-open cover for X. Since $U_i \in S$ for every $i \in I$, $U_i \in T[S,T]$ for every $i \in I$. But $(X,T[S,T])$ is compact, so A must have a finite subcover $U_1, U_2, \ldots, U_n$ for some positive integer n. Therefore (X,S) is compact.

EXAMPLE 15. Let $X = (0,1)$ and define topologies S and T for X as follows: $S = (\psi - \{U \in \psi \mid 1/4 \in U\}) \cup \{X\}$ and $T = (\psi - \{U \in \psi \mid 3/4 \in U\}) \cup \{X\}$ where ψ is the usual topology for X. Then (X,S) and (X,T) are compact but $(X,T[S,T])$ is not.

PROOF: Since X is the only S-open set containing the point $1/4$, any S-open cover for X must contain X. Therefore $\{X\}$ is a finite subcover for any open cover of X and (X,S) is compact.

Similarly, it can be shown that (X,T) is compact.

It is evident that the smallest topology containing both S and T is the usual topology so $T[S,T] = \psi$. But (X,ψ) is not compact since $A = \{(0, (2n - 1)/2n) \mid n = 1, 2, \ldots\}$ is a ψ-open cover for X which has no finite subcover.

DEFINITION 30. A topological space X is countably compact if and only if each countable open cover of X has a finite subcover.

THEOREM 11. Let X be a set with topologies S and T. If $(X,T[S,T])$ is countably compact then (X,S) is countably compact.

PROOF: Let $A = \{U_n \mid n = 1, 2, 3, \ldots\}$ be a countable S-open cover for X. Clearly A is also a $T[S,T]$-open cover and thus has a finite subcover $U_1, U_2, \ldots, U_n$. Therefore (X,S) is countably compact.

EXAMPLE 16. Let $X = (0,1)$ and let the topologies S and T be those defined in example 15. Then (X,S) and (X,T) are countably compact, but $(X,T[S,T])$ is not.

PROOF: Since (X,S) and (X,T) are compact they are clearly countably compact. Let $A = \{(0,(2n - 1)/2n) \mid n = 1, 2, \ldots\}$ be a $T[S,T]$-open cover for X. The collection A is countable since each $(2n - 1)/2n$ is a rational number. However, it was shown in example 15 that A has no finite subcover. Therefore $(X,T[S,T])$ is not countably compact.

DEFINITION 31. A topological space (X,S) is Lindelof provided every open cover of X has a countable subcover.

<u>THEOREM 12</u>. Let X be a set with topologies S and T. If $(X,T[S,T])$ is Lindelof, then (X,S) is Lindelof.

<u>PROOF:</u> Clearly any S-open cover A of X is also a $T[S,T]$-open cover. Since $(X,T[S,T])$ is Lindelof, A has a countable subcover. Therefore (X,S) is Lindelof.

<u>EXAMPLE 17</u>. Let $X = [-1,1]$ and define topologies S and T for X as follows: a set U is in T provided $U = X$ or U does not contain the point $1/2$, and a set V is in S provided either V does not contain the point 0 or does contain the interval $(-1,1)$. Then (X,S) and (X,T) are Lindelof, but $(X,T[S,T])$ is not.

<u>PROOF:</u> Let A be a T-open cover for X. Then $X \in A$ since X is the only T-open set containing $1/2$. Therefore $\{X\}$ is a countable subcover for A and (X,T) is Lindelof.

Now let A' be an S-open cover for X. Then there must be a set $U \in A'$ such that $0 \in U$. If $0 \in U$ then $(-1,1) \subset U$ by definition of S. There must also be a set V_1 such that $\{-1\} \subset V_1$ and a set V_2 such that $\{1\} \subset V_2$. Then $\{U, V_1, V_2\}$ is a countable subcover for A' so (X,S) is Lindelof.

Let $x \in X$. If $x \neq 0 \neq 1/2$ let $t = \{x\}$ and let $s = \{x\}$. Then $s \in S$ and $t \in T$ and $x \in s \cap t \subset \{x\}$ so $\{x\} \in T[S,T]$. If $x = 0$, let $t = \{0\}$ and let $s = X$. Then $s \in S$ and $t \in T$ and $x \in s \cap t \subset \{x\}$. Finally if $x = 1/2$, let $t = X$ and $s = \{1/2\}$ and again we find that $\{x\} \in T[S,T]$. Therefore $T[S,T]$ is the discrete topology for X. Furthermore $(X,T[S,T])$ is not Lindelof since $A = \{\{x\} \mid x \in X\}$ is an open cover for X which has no countable subcover.

DEFINITION 32. A topological space X is connected if and only if there are no two disjoint nonempty open sets H and K in X such that $X = H \cup K$.

THEOREM 13. Let X be a set and let S and T be topologies for X such that $(X,T[S,T])$ is connected. Then (X,S) is connected.

PROOF: Suppose (X,S) is not connected. Then there are disjoint nonempty S-open sets H and K such that $X = H \cup K$. But if H and K are in S, they are also in $T[S,T]$ and so $(X,T[S,T])$ is not connected. This is a contradiction and therefore (X,S) is connected.

EXAMPLE 18. Let X be the real numbers and let T be the cofinite topology for X. Let $p \in X$ and let S be the particular point topology with respect to p. Then (X,S) and (X,T) are connected, but $(X,T[S,T])$ is not.

PROOF: Clearly the spaces (X,S) and (X,T) are connected because no two open sets are disjoint. By example 1 $T[S,T]$ is the discrete topology. Let $x \in X$. Then $\{x\}$ and $X - \{x\}$ are two disjoint nonempty $T[S,T]$-open sets whose union is X. Therefore $(X,T[S,T])$ is not connected.

DEFINITION 33. A topological space X is locally connected provided each $x \in X$ has a neighborhood base of open connected sets.

<u>EXAMPLE 19</u>. Let X be the real numbers and define topologies S and T for X as follows: a set U is open in S provided $U = X$ or the complement of U contains the point 1; and a set V is open in T provided the complement of T is finite. Then (X,S) and (X,T) are locally connected, but $(X,T[S,T])$ is not.

<u>PROOF</u>: Let $x \in X$ and let $U \in S$ such that $x \in U$. If $x \neq 1$, then $\{x\}$ is a connected set containing x and contained in U. If $x = 1$ it is clear that $U = X$. Now suppose there are disjoint nonempty S-open sets M and N such that $M \cup N = X$. Then $1 \in M \cup N$ so either $1 \in M$ or $1 \in N$. It follows that $M = X$ or $N = X$. Therefore M and N are not disjoint. This is a contradiction, so X is connected. We have shown that x has a neighborhood base of S-open connected sets and so (X,S) is locally connected.

Now let $B = \{U_\alpha \mid \alpha \in A\}$ be a neighborhood base for x of T-open sets and let $V \in T$ such that $x \in V$. Then there is an $\alpha_1 \in A$ such that $U_{\alpha_1} \in B$ and $x \in U_{\alpha_1} \subset V$. Since there are no disjoint T-open sets, U_{α_1} must be connected and therefore (X,T) is locally connected.

Evidently $T[S,T]$ is the topology in which a set U is open provided the complement of U is either finite or includes the point 1. Now consider the point 1 and the $T[S,T]$-open set $U = (-\infty,-2) \cup (-2,-1) \cup (-1,1) \cup [1,3) \cup (3,\infty)$. Clearly any $T[S,T]$-open set containing 1 and contained in U is not connected so $(X,T[S,T])$ is not locally connected.

DEFINITION 34. If (a,b) and (c,d) are in the plane then $d((a,b),(c,d)) = \sqrt{(a-c)^2 + (b-d)^2}$.

DEFINITION 35. Let X be the plane. Then the usual topology for X is defined as follows: a subset O of the plane is open provided if $p \in O$ then there exists an $r > 0$ such that $\{x \mid d(x,p) < r\} \subset O$.

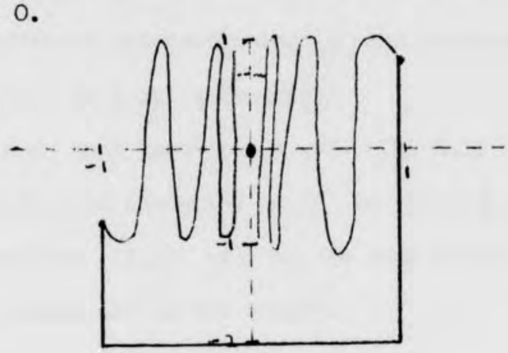

FIGURE 1

EXAMPLE 20. Let $X = \{(x,\sin(1/x)) \mid x \in [-1,0) \cup (0,1]\} \cup \{(0,0)\} \cup \{(-1,y) \mid y \in [-2,\sin(-1)]\} \cup \{(x,-2) \mid x \in [-1,1]\} \cup \{(1,y) \mid y \in [-2,\sin(1)]\}$ and define topologies S and T for X as follows: S and T are the relative topologies induced from the usual topology on the plane at every point except $(0,0)$. A basic S-open set about $(0,0)$ is of the form $\{(x,y) \mid x^2 + y^2 \leq r^2 \text{ for some real number } r > 0 \text{ and } x \geq 0\} \cap \{X\}$, and a basic T-open set about $(0,0)$ is of the form $\{(x,y) \mid x^2 + y^2 \leq r^2 \text{ for some real number } r > 0 \text{ and } x \leq 0\} \cap \{X\}$. Then $(X,T[S,T])$ is locally connected, but (X,S) and (X,T) are not.

<u>PROOF</u>: Clearly $T[S,T]$ is the union of the relative topology for X with the set $\{(0,0)\}$. Now let $p \in X$ such that $p \neq (0,0)$. An open set U about p is the intersection of a disc about p with X. By taking a second disc about p with an appropriately small radius we obtain an open connected set containing p and contained in U. If $p = (0,0)$ and $U \in T[S,T]$ such that $p \in U$, $\{p\}$ is an open connected set containing p and contained in U. Therefore $(X,T[S,T])$ is locally connected.

Now let $U \in S$ such that $(0,0) \in U$. The only connected set containing $(0,0)$ and contained in U is $\{(0,0)\}$, but $\{(0,0)\}$ is not in S. Therefore (X,S) and, in the same manner, (X,T) fail to be locally connected at the origin.

SUMMARY

In this paper we have examined the upper bound topology $T[S,T]$ and its relation to the topologies S and T. The separation axioms, countability, compactness, and connectedness have all been investigated.

Some interesting questions arose which were left unanswered. If (X,S) and (X,T) are regular (normal, hereditarily separable) is $(X,T[S,T])$ necessarily regular (normal, hereditarily separable)? If $(X,T[S,T])$ is hereditarily separable is (X,S) hereditarily separable?

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